ON THE PARAMETRIC INSTABILITY OF A RIGID ROTATION OF A FLUID

(O PARAMETRICHESKOI NEUSTOICHIVOSTI TVERDOGO VRASHCHENIIA ZHIDKOSTI)

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The problem concerning the convective stability of a nonstationary equilibrium of the fluid was solved previously [1]. In that paper the vertical temperature gradient, which corresponds to nonstationary equilibrium, is modulated periodically with certain frequency and amplitude. As it was shown in [1], the modulation of the parameter influences considerably the convective stability. Under certain conditions this influence turns out to be stabilizing: the modulation of the temperature gradient increases the stability of equilibrium in comparison with the stationary case. A similar effect of the parametric increase of stability was detected by experiment [2]. There was observed the appearance of Taylor's instability of fluid motion between the cylinders with the modulation of angular velocity.

The present paper investigates the stability of the most simple nonstationary motion of the fluid which is the modulated rigid rotation. As it is known (see [3 and 4]), the stationary rotation of the fluid as a rigid body is stable with respect to the small perturbations for all the values of the angular velocity of rotation. The modulation of angular velocity leads to the appearance of unstable regions. The influence of the modulation of the parameter turns out to be destabilizing.

1. Let us assume that the fluid fills completely the cavity which has the shape of a body of revolution. The boundaries of the cavity rotate around a fixed axis with angular velocity, which varies periodically with time

$$\Omega(t) = \Omega_1 + \Omega_2 \sin \omega_0 t \tag{1.1}$$

where Ω_1 is the average angular velocity of rotation, and Ω_2 is the amplitude of modulation. Let us consider the case of a slow modulation, when the frquency ω_0 is small, i.e.

$$\omega_0 \ll \nu / L^2 \tag{1.2}$$

Here ν is the kinematic viscosity, and L is the characteristic dimension. In this case the quasi-stationary approximation is valid: the fluid will rotate as a rigid body with a uniform angular velocity (1.1), equal to the velocity of the boundaries. In a stationary system of coordinates the velocity of the fluid is

$$\mathbf{v}_0 = \mathbf{\Omega}(t) \times \mathbf{r} \tag{1.3}$$

Let us consider small perturbations of a nonstationary rotation of the fluid (1.3). In the system of coordinates, which rotates with the fluid with angular velocity $\Omega(t)$, the equations of small perturbations have the form

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{1}{\rho}\nabla p + \nu\Delta\mathbf{v} - 2(\Omega\times\mathbf{v}), \qquad \operatorname{div}\mathbf{v} = 0 \tag{1.4}$$

where $\mathbf{v}$ and p are the perturbations of the velocity and of pressure. (In the above equation the centrifugal force $\Omega\times(\mathbf{r}\times\Omega)$ and the inertia force, caused by the nonuniform fotation $\mathbf{r}\times\dot{\Omega}$ of the system of coordinates, drop out on account of the equation for the unperturbed motion).

2. Let us consider at first an infinite layer of fluid, bounded by the planes $z = \pm h$ and rotating about the z-axis. Applying the operations curl and curl curl to Equation (1.4) abd projecting the resulting equations on the axis of rotation z, we obtain a system of equations for the z-components of the velocity v_z and for the curl of the velocity $F = \mathrm{curl}_z\mathbf{v}$

$$\frac{\partial F}{\partial t} - 2\Omega(t)\frac{\partial v_z}{\partial z} = \nu\Delta F, \qquad \frac{\partial}{\partial t}\Delta v_z + 2\Omega(t)\frac{\partial F}{\partial z} = \nu\Delta\Delta v_z \tag{2.1}$$

From the system (2.1) we can eliminate F and obtain an equation which contains only v_z. With the assumption (1.2) it reduces to the form

$$\left(\frac{\partial}{\partial t} - \nu\Delta\right)^2\Delta v_z = -4\Omega^2(t)\frac{\partial^2 v_z}{\partial z^2} \tag{2.2}$$

Equation (2.2) admits a simple exact solution in the case, when the velocity perturbations satisfy the conditions on the free boundaries given by

$$v_z = v_z'' = v_z'''' = 0 \qquad \text{for } z = \pm h \tag{2.3}$$

(the differentiation with respect to z is indicated by a prime). Let us take

$$v_z = v(t)\exp[i(k_1x + k_2y)]\begin{cases}\cos[{}^1/_2(n+1)\pi z/h] & (n = 0, 2, 4, \ldots)\\ \sin[{}^1/_2(n+1)\pi z/h] & (n = 1, 3, 5, \ldots)\end{cases} \tag{2.4}$$

Substituting (2.4) into (2.2) we obtain

$$v^{\cdot\cdot} + 2\nu\varkappa^2 v^{\cdot} + \nu^2\varkappa^4\left[1 + \frac{(n+1)^2\pi^2}{\nu^2\varkappa^6h^2}\Omega^2(t)\right]v = 0 \tag{2.5}$$

$$\varkappa^2 = k_1^2 + k_2^2 + {}^1/_4(n+1)^2\pi^2/h^2$$

By choosing as a unit of time the magnitude $1/\nu\varkappa^2$, we can reduce Equation (2.5) to the form

$$v^{\cdot\cdot} + 2v^{\cdot} + [1 + (T_1 + T_2\sin p_*t)^2]\,v = 0 \tag{2.6}$$

$$T_1 = \frac{(n+1)\pi}{\nu\varkappa^3h}\Omega_1, \qquad T_2 = \frac{(n+1)\pi}{\nu\varkappa^3h}\Omega_2, \qquad p_* = \frac{\omega_0}{\nu\varkappa^2} \tag{2.7}$$

Here p_* is the nondimensional frequency modulation, T_1 and T_2 are the nondimensional parameters, which define the average angular velocity and the amplitude of modulation; the corresponding Taylor numbers are equal to T_1^2 and T_2^2.

Let us now consider the two-dimensional plane infinite layer with rigid boundaries. In this case the boundary conditions are

$$v_z = v_z' = F = 0 \qquad \text{for} \quad z = \pm h \tag{2.8}$$

With these boundary conditions we can obtain an effective approximate solution by the Galerkin method. Let us assume

$$v_z = \exp\,[i\,(k_1x + k_2y)] \sum_i v_i\,(t)\, p_i\,(z), \qquad F = \exp\,[i\,(k_1x + k_2y)] \sum_i f_i\,(t)\, q_i(z) \tag{2.9}$$

where the functions $p_i(z)$ and $q_i(z)$ satisfy the boundary conditions

$$p_i = p_i' = q_i = 0 \qquad \text{for} \quad z = \pm h \tag{2.10}$$

The functions $v_i(t)$ and $f_i(t)$ in (2.9) are the coefficients of expansion of v_z and F in eigenfunctions. These coefficients are determined from the usual conditions of the Galerkin method. In this way we obtain a system of first order differential equations for $v_i(t)$ and $f_i(t)$ with periodic coefficients.

The system (2.1) has solutions of "even" and "odd" type. The even function $v_z(z)$ and the odd function $F(z)$ correspond to the even solution and vice versa for the "odd" solution. There exists an infinite sequence of solutions of each type. We will be interested only in the first solutions of these sequences (in the case of free boundaries the numbers $n = 0,1$ correspond to these first solutions in Equation (2.4)).

Let us examine the "even" solution. Leaving in the first approximation only one of the terms in the summations (2.9), we select for the functions $p(z)$ and $q(z)$ polynomials, which satisfy the conditions (2.10) and an additional condition $q''(\pm h) = 0$, which follows from Equation (2.1)

$$p\,(z) = (1 - \zeta^2)^2, \qquad q\,(z) = \zeta\,(1 - \zeta^2)\,(7 - 3\zeta^2) \qquad (\zeta = z\,/\,h) \tag{2.11}$$

The Galerkin conditions lead to the system of equations for $v(t)$ and $f(t)$. Eliminating from it $f(t)$ we obtain

$$v^{\cdot\cdot} + 2\varepsilon v^{\cdot} + [1 + (T_1 + T_2 \sin p_* t)^2]\, v = 0 \tag{2.12}$$

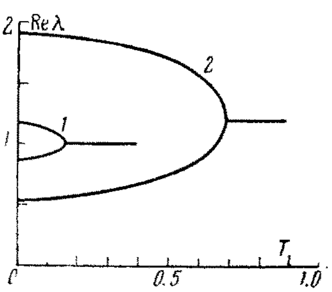

Fig. 1

The unit of time is chosen as $\frac{1}{\nu \varkappa^2}$, where

$$\left(\varkappa^2 = \frac{99 + 10k^2}{10h^2 \sqrt{P}}\,,\ k^2 = (k_1^2 + k_2^2)\, h^2\right)$$

and the following notations are introduced

$$P = \frac{(3 + k^2)\,(99 + 10k^2)}{5\,(63 + 12k^2 + 2k^4)}\,, \qquad 2\varepsilon = \frac{1 + P}{\sqrt{P}}$$

$$T_{1,2} = \frac{3}{5\nu\varkappa^2}\left(\frac{33}{3 + k^2}\right)^{1/2} \Omega_{1,2}$$

In order to find the first "odd" solution we choose the polynomials

$$p\,(z) = \zeta\,(1 - \zeta^2)^2, \qquad q\,(z) = (1 - \zeta^2)\,(5 - \zeta^2) \tag{2.13}$$

The Galerkin method leads again to the equation of the type (2.12), where we have now

$$\varkappa^2 = \frac{153 + 62k^2}{62h^2 \sqrt{P}}, \quad P = \frac{(153 + 62k^2)(11 + k^2)}{31(495 + 44k^2 + 2k^4)}, \quad T_{1,2} = \frac{4}{\nu\varkappa^2}\left[\frac{11}{31(11 + k^2)}\right]^{1/2} \Omega_{1,2}$$

Equation (2.12) differs from (2.6) only in the value of the damping coefficient ε. In the case of the free boundaries $\varepsilon = 1$; in the case of rigid boundaries $\varepsilon \geqslant 1$ and the equation depends on the wave number k.

In this way, the investigation of the limits of stability reduces to the determination of the conditions of existence of the periodic solutions of Equation (2.12)-

3. Let us consider the uniform rotation without modulation $(T_2 = 0)$. Then the behavior of the perturbations is determined by the equation with constant coefficients, for which exist particular solutions of the type $e^{-\lambda t}$, where

$$\lambda_{\pm} = \varepsilon \pm \sqrt{\varepsilon^2 - 1 - T_1^2} = \frac{1 + P}{2\sqrt{P}} \pm \left(\frac{(1 - P)^2}{4P} - T_1^2\right)^{1/2} \qquad (3.1)$$

From this it follows, that always $\operatorname{Re} \lambda_{\pm} > 0$, i.e. the perturbations damp out for all the values of T_1. If $T_1 < T_{1*}$, where

$$T_{1*}^2 = 1/4\, P^{-1} (1 - P)^2,$$

then both decrements are real, i.e. the perturbations damp out monotonously, and for $T_1 = T_{1*}$ both decrements coincide: $\lambda_+ = \lambda_-$. For $T_1 > T_{1*}$ the decrements λ_+ and λ_- become complex conjugate: the perturbations damp out, oscillating with the frequency $\pm (T_1^2 - T_{1*}^2)^{1/2}$.

In Fig.1 are represented, as an illustration, four lower decrements for $k = 2$ which depend on T_1. The curves 1 and 2 correspond to the even and odd solutions, respectively. For $T_1 = 0$ the values of the decrements do not differ practically from the exact values, which in this case are easily determined. In the case of free boundaries $\varepsilon = 1$ and $T_{1*} = 0$, i.e. the perturbations have already an oscillatory character for an arbitrarily small velocity of rotation (because in the case of free boundaries the spectrum of decrements $\lambda_{\pm}$ for $T_1 = 0$ turns out to be degenerate).

4. During the construction of the stable regions of Equation (2.12), in the same way as in [1], we shall replace the sinusoidal modulation by a rectangular one, i.e. instead of the modilating function $\sin p_* t$ we shall consider a periodic function, which takes constant values ± 1 every half-period. Then we can write the general solution of Equation (2.12) for every half-period. The conditions of continuity and periodicity of the function and its derivative determine a nontrivial periodic solution (2.12), if between the parameters of the equation is satisfied the following relation

$$\begin{gathered}\cos\frac{\alpha}{p}\cos\frac{\beta}{p} - \frac{\alpha^2 + \beta^2}{2\alpha\beta}\sin\frac{\alpha}{p}\sin\frac{\beta}{p} = \pm\cosh\frac{2\varepsilon}{p} \qquad \left(p = \frac{p_*}{\pi}\right) \\ \alpha = \sqrt{1 - \varepsilon^2 + (T_1 + T_2)^2}, \qquad \beta = \sqrt{1 - \varepsilon^2 + (T_1 - T_2)^2}\end{gathered} \qquad (4.1)$$

The relation (4.1) allows, for example, for fixed values of the average velocity of rotation and the wave number (i.e. for fixed T_1 and ε) to find the region of stable and unstable values of the amplitude and the frequency of modulation T_2 and p. Equation (4.1) defines the boundaries of these regions (it is convenient to consider these regions on the coordinate plane $\sqrt{T_2}$, p^{-1}; the sign "plus" and "minus" on the right-hand side of (4.1) correspond to the integer and "half-integer" periodic solutions).

Let us consider the unstable regions for $\epsilon = 1$. In this case there exists a threshold value of the average velocity of rotation. If $T_1 < \frac{1}{2}\pi$, then the rotation is stable for all the frequencies and amplitudes of modulation. The unstable regions appear for $T_1 > \frac{1}{2}\pi$ (as an example, in Fig.2 are represented these regions for $T_1 = 6$; the shaded regions are unstable). With the increasing value of T_1 (i.e. the average velocity of rotation) the number of unstable regions increases. These regions are oriented in the direction of the asymptotic lines

$$2T_2/p = m\pi \qquad (m = 1, 2, 3\ldots) \tag{4.2}$$

(for all odd values of m corresponds the "minus" sign in (4.1), and vice versa for the even values of m). Every region is engendered for a definite threshold value of T_1 as a point on the plane (T_2, p).

For the threshold value $T_1 = \frac{1}{2}\pi$ the region $m = 1$ is engendered (the extreme one to the left in Fig.2); the coordinates of the points of engendration are: $1/p = 0$, $T_2 = \infty$.

Let us give also the threshold values of T_1 and the coordinates of the points of engendration of the two following unstable regions

$$m = 2\ \{T_1 = 2.7,\ 1/p = 0.70,\ T_2 = 4.2\}$$
$$m = 3\ \{T_1 = 3.6;\ 1/p = 0.97,\ T_2 = 4.5\}$$

For big values of m the new unstable regions appear for $T_2 \approx T_1$; the threshold parameters are determined by the relations

$$\cosh\frac{2}{p} = (2m-1)\frac{\pi}{4}, \qquad T_1 = p\cosh\frac{2}{p} \tag{4.3}$$

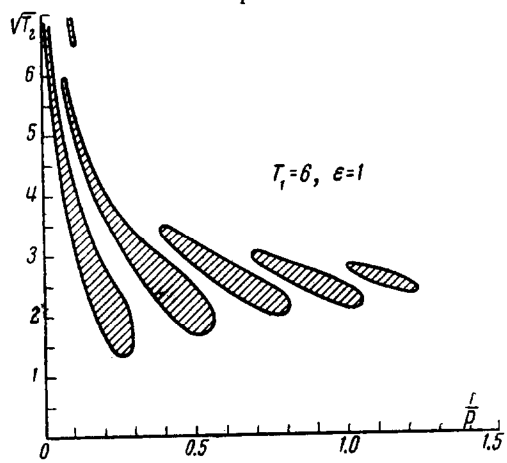

Fig. 2

In this way, for big values of T_1 there exists an unstable strip, which consists of alternating regions. The resonance value of the amplitude $T_2 = T_1$. The low-frequency boundary of the unstable strip is determined by the relations (4.3), from which we see, that the maximum value of $1/p$ increases monotonously with the increase of T_1; i.e. for a fast rotation the parametric instability can be aroused by the low-frequency modulations. There exists an absolute threshold for the amplitude: for $T_2 < \frac{1}{2}\pi$ the rotation is stable for all values of T_1.

Besides the indicated unstable regions which form the basic strip, there exist also narrow regions, lying on the same asymptotic lines (4.2) but corresponding to the larger values of the amplitude T_2. These regions appear for larger values of T_1, then the corresponding regions of the lower strip. The number of these regions increases with m. One of these regions ($m = 3$) is shown in Fig.2.

The increase of the friction parameter ε leads, as it must be expected, to the increase of stability. The unstable regions are displaced in the direction of the larger amplitudes T_2; the appearance of the new unstable regions occurs for larger values of T_1. Let us give the formulas for the threshold value of the parameter T_1 and for the resonance value of the amplitude T_2, which determines the position of the basic unstable strip

$$T_1 = {}^1/_2\,\pi \sqrt{\varepsilon^2 + 2\pi^{-2}(\varepsilon^2 - 1)}, \qquad T_2 = T_1 + \sqrt{\varepsilon^2 - 1} \tag{4.4}$$

The discussed singularities of the unstable spectrum, apparently, are conserved also in the case of a sinusoidal modulation. The unstable regions in this case can be found by the Fourier method. For the first region ($m=1$) of parameteric resonance we have

$$v = A_1 \sin {}^1/_2 p_* t + B_1 \cos {}^1/_2 p_* t + A_3 \sin {}^3/_2 p_* t + B_3 \cos {}^3/_2 p_* t + \ldots \tag{4.5}$$

Limiting ourselves to the basic harmonic frequency $\frac{1}{2}p_*$, we obtain for the boundary of the region the following equation:

$$T_2^2 = {}^1/_2\, p_*^2 - 2 \pm \sqrt{2p_*^2 (T_1^2 - 2\varepsilon^2) - 4T_1^4 - 8T_1^2} \tag{4.6}$$

From (4.6) we find the threshold value $T_1 = \varepsilon/2$. Hence for $\varepsilon = 1$ we obtain $T_1 = \sqrt{2}$, instead of $T_1 = \frac{1}{2}\pi$ in the case of rectangular modulation.

5. The method of computation, applied in Section 2 to the two-dimensional plane layer with rigid boundaries, can be used for the investigation of the parametric perturbation of the rigid rotation in the cavities of the other shapes. The first approximations of the Galerkin method lead to Equation (2.12) with the corresponding values of the parameters. Let us give the results of calculation for a cylinder and a thin cylindrical layer, rotating with respect to an axis (the thin cylindrical layer, apparently, is more interesting from the experimental point of view).

Considering the perturbations axially-symmertical and depending on the coordinate z according to the law e^{ik_1z}, we obtain for the components v_r of the velocity and for the components v_φ of the perturbation the following equations:

$$Dv_r^{\cdot} + 2\Omega(t)\, k_1^2\, v_\varphi = \nu D^2 v_r \qquad \left(D = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} - \frac{1}{r^2} - k_1^2\right) \tag{5.1}$$

$$v_\varphi^{\cdot} + 2\Omega(t)\, v_r = \nu D v_\varphi \tag{5.2}$$

(r, φ and z are the cylindrical coordinates).

On the rigid boundary $v_r = v_r' = v_\varphi = 0$. During approximation it is useful to take into account an additional condition $Dv_\varphi = 0$, which follows from (5.2).

In the case of a cylinder of radius R the velocities are approximated in the following way:

$$v_r = v(t)\,\zeta(1-\zeta^2)^2, \quad v_\varphi = f(t)\,(1-\zeta^2)(2\zeta-\zeta^3) \qquad (\zeta = r/R) \tag{5.3}$$

For the function $v(t)$ we obtain Equation (2.12) with the parameters given by

$$\varkappa^2 = \frac{440+31k^2}{31R^2\sqrt{P}}, \qquad k = k_1R$$

$$P = \frac{(33+2k^2)(440+31k^2)}{62(396+33k^2+k^4)}, \quad T_{1,2} = \frac{38k\Omega_{1,2}}{\nu\varkappa^2[186(33+2k^2)]^{1/2}}$$

Let us consider also a thin cylindrical layer of thickness $2h = R_2 - R_1$; the inner and outer radii R_1 and R_2 are close to each other, $h \ll R_1$. In the operator D, contained in (5.1) and (5.2), we can neglect the curvature

$$D = \frac{\partial^2}{\partial r^2} - k_1^2$$

Let us introduce a nondimensional coordinate ζ, measured from the middle of the layer

$$\zeta = h^{-1}\left[r - \tfrac{1}{2}(R_1+R_2)\right]$$

and let us approximate the components of the velocity (the even perturbation) by Formulas

$$v_r = v(t)(1-\zeta^2)^2, \qquad v_\varphi = f(t)(1-\zeta^2)(5-\zeta^2) \tag{5.4}$$

In this case the parameters of Equation (2.12) are defined by Formulas

$$\varkappa^2 = \frac{153+62k^2}{62h^2\sqrt{P}}, \quad P = \frac{(3+k^2)(153+62k^2)}{31(63+12k^2+2k^4)}, \quad T_{1,2} = \frac{11k\Omega_{1,2}}{\nu\varkappa^2[31(3+k^2)]^{1/2}}, \quad k = k_1h$$

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